

Added mass partitioned fluid-structure interaction solver based on Robin boundary condition for pressure

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Introduction

- ▶ Added mass effect (instability) occurs when loosely coupled partitioned schemes are used for FSI cases with solid to fluid density ratio close to 1
- ▶ Even strongly coupled Dirichlet-Neumann (D-N) partitioned schemes need some kind of stabilization procedures in order to deal with the cases characterized by added-mass effect
- ▶ Stabilization is usually done by fixed or dynamic (Aitken) relaxation or by more advanced schemes like Degroote's IQN-ILS scheme
- ▶ Loosely coupled partitioned schemes which can efficiently cope with the added mass effect are very rare
- ▶ One of such scheme is the so called kinematically coupled β -scheme based on the Lie operator splitting and Robin-type boundary condition for fluid
- ▶ Using idea of kinematically coupled β -scheme, Robin type boundary condition for pressure is proposed which allows inclusion of the structure inertia into the fluid sub-problem

Kinematically coupled β -scheme

Fluid sub-problem [inviscid Stokes flow (pressure formulation)]:

$$-\nabla \cdot \nabla p^{n+1/2} = 0$$

$$\boxed{p^{n+1/2} + \frac{\rho_s h_s}{\rho_f} \left(\frac{\partial p}{\partial n} \right)^{n+1/2}} = \beta p^n$$

$$\frac{\boxed{v_r^{n+1/2}} - v_r^n}{\Delta t} = -\frac{1}{\rho_f} \left(\frac{\partial p}{\partial n} \right)^{n+1/2}$$

Solid sub-problem (thin elastic structure):

$$\rho_s h_s \frac{1}{\Delta t} \left[\left(\frac{\partial u_r}{\partial t} \right)^{n+1} - \boxed{v_r^{n+1/2}} \right] + L(u_r^{n+1}) = \beta p^{n+1/2}$$

Mathematical Model for Fluid Flow

Incompressible Navier-Stokes equations (laminar flow)

$$\oint_S \mathbf{n} \cdot \mathbf{v} \, dS = 0$$

$$\frac{d}{dt} \int_V \mathbf{v} \, dV + \oint_S \mathbf{n} \cdot \rho (\mathbf{v} - \mathbf{v}_s) \mathbf{v} \, dS = \oint_S \mathbf{n} \cdot (\nu \nabla \mathbf{v}) \, dS - \frac{1}{\rho} \int_V \nabla p \, dV$$

Space conservation law:

$$\frac{d}{dt} \int_V dV - \oint_S \mathbf{n} \cdot \mathbf{v}_s \, dS = 0.$$

Boundary condition for pressure at moving walls (FSI interface)

$$\mathbf{n} \cdot \nabla p = -\rho_f \frac{\partial(\mathbf{n} \cdot \mathbf{v})}{\partial t} = -\rho_f \frac{\partial v_n}{\partial t}$$

Fluid Mesh Motion/Deformation Model

- ▶ Finite volume mesh in the fluid domain must adjust itself to the time varying shape of the interface
- ▶ Mesh deformation approach is used: internal finite volume (cell) vertices are moved based on the prescribed motion of the boundary vertices
- ▶ Mesh deformation is governed by Laplace equation for displacement with variable diffusion coefficient

$$\nabla \cdot (\gamma \nabla \mathbf{u}) = 0$$

- ▶ To make mesh more rigid near the moving/deforming boundaries (interface), diffusing coefficient γ is inversely proportional to the distance from the selected boundaries
- ▶ Mesh motion equation discretized using cell-centered finite volume method (calculated cell-center displacement is interpolated to the cell vertices)

Mathematical Model for Solid Deformation

Momentum equation in total Lagrangian formulation

$$\int_{V_0} \rho_0 \frac{\partial}{\partial t} \left(\frac{\partial \mathbf{u}}{\partial t} \right) dV = \int_{S_0} \mathbf{n} \cdot (\boldsymbol{\Sigma} \cdot \mathbf{F}^T) dS$$

St. Venant-Kirchhoff constitutive model:

$$\boldsymbol{\Sigma} = 2\mu \mathbf{E} + \lambda \operatorname{tr}(\mathbf{E}) \mathbf{I}$$

Green-Lagrangian strain tensor:

$$\mathbf{E} = \frac{1}{2} \left[\nabla \mathbf{u} + (\nabla \mathbf{u})^T + \nabla \mathbf{u} \cdot (\nabla \mathbf{u})^T \right]$$

Relation between Cauchy and second Piola-Kirchhoff stress tensor

$$\boldsymbol{\sigma} = \frac{1}{\det \mathbf{F}} \mathbf{F} \cdot \boldsymbol{\Sigma} \cdot \mathbf{F}^T$$

Conditions at the fluid-structure interface

Kinematic condition (continuity of velocity and displacement)

$$\mathbf{v}_{f,i} = \mathbf{v}_{s,i},$$

$$\mathbf{u}_{f,i} = \mathbf{u}_{s,i}$$

Dynamic condition (equilibrium of forces)

$$\mathbf{n}_i \cdot \boldsymbol{\sigma}_{f,i} = \mathbf{n}_i \cdot \boldsymbol{\sigma}_{s,i},$$

Discretisation of fluid and solid model

- ▶ Discretization in space using cell-centred second order accurate finite volume method, where mid-point rule is used to approximate volume and surface integrals
- ▶ Diffusion terms discretised using central scheme, and convection terms using linear upwind scheme
- ▶ Cell-centre gradients and face-centre gradients in tangential direction calculated using point-Gauss method
- ▶ Discretisation in time using first order accurate implicit Euler scheme
- ▶ Both fluid and solid models solved using segregated solution procedure
- ▶ PISO procedure used to ensure coupling between pressure and velocity in the fluid model

Dirichlet-Neumann strongly coupled partitioned scheme

Solution procedure:

- 1: Predict interface displacement by solving solid model using fluid force at the interface from the previous time step
- 2: Solve mesh motion equation with specified interface displacement, move fluid mesh and calculate **interface velocity and acceleration**
- 3: Solve fluid model on deformed fluid mesh using **Neumann BC for pressure** and **Dirichlet BC for velocity** at the interface
- 4: Transpose fluid force from the fluid to the solid side of the interface and solve the solid model using **Neumann BC for displacement** at the interface
- 5: Transpose displacements from the solid to the fluid side of the interface and calculate interface residual \mathbf{r}
- 6: **if** $|\mathbf{r}| > \epsilon$ **then**
| Calculate new fluid side interface displacement (using fixed relaxation, Aitken or IQN-ILS) and return to step 2
end

New Robin-Neumann strongly coupled scheme

- ▶ Proposal: Use Robin (instead Neumann) type boundary condition for pressure at the interface in order to include solid inertia into the solution of fluid model
- ▶ Neumann boundary condition for pressure used in the Dirichlet-Neumann coupling procedure is defined as:

$$\frac{\partial p}{\partial n} = -\rho_f \frac{\partial v_n}{\partial t} \quad (1)$$

where temporal derivative of the normal component of velocity is calculated from the motion of the solid side of the interface. On the other hand motion of the solid side of the interface is driven by the fluid pressure from the previous iteration

- ▶ This approach is unstable and stabilisation (or convergence acceleration) schemes are required (Aitken, IQN-ILS)

New Robin-Neumann strongly coupled scheme

- ▶ Using the idea of kinematically coupled β -scheme one can state that fluid pressure at the interface is limited (or can be approximated) by the solid inertia:

$$p \approx \rho_s h_{sn} \frac{\partial v_n}{\partial t}, \quad (2)$$

where h_{sn} is thickness of the structure (procedure to calculate this thickness in case of thick structure will be defined later)

- ▶ Combining two previous equations through $\frac{\partial v_n}{\partial t}$ term, one can derive Robin type boundary condition for pressure:

$$p + \frac{\rho_s h_{sn}}{\rho_f} \frac{\partial p}{\partial n} = 0 \quad (3)$$

New Robin-Neumann strongly coupled scheme

- ▶ Robin boundary condition defined by previous equation is not fully correct because $p \approx \rho_s h_{sn} \frac{\partial v_n}{\partial t}$ is only approximation
- ▶ Taking into account that iterative solution procedure will be used (strong coupling), Robin boundary condition can be formulated in such a way so that it reduces to exact BC for pressure when converged solution is reached:

$$p^k + \frac{\rho_s h_{sn}}{\rho_f} \left(\frac{\partial p}{\partial n} \right)^k = p^{(k-1)} - \rho_s h_{sn} \left(\frac{\partial v_n}{\partial t} \right)^{(k-1)} \quad (4)$$

where k is iteration counter.

- ▶ When converged solution is reached ($p^k = p^{(k-1)}$), above equation will be reduced to $\frac{\partial p}{\partial n} = -\rho_f \frac{\partial v_n}{\partial t}$
- ▶ Converged solution will not depend on the parameter h_{sn} . Only stability could depend on it.

New Robin-Neumann strongly coupled scheme

- ▶ In the case when shell model is used to define structure deformation, the thickness parameter h_{sn} is specified value
- ▶ It is not clear how to calculate h_{sn} in case when thick structure model is used
- ▶ Possible approach is to use propagation speed of waves in elastic media (J.W. Banks et.al., JCP 269 (2014) 108–137)

$$h_{sn} = a_p \Delta t \quad (5)$$

where a_p is propagation speed of p-wave defined by

$$a_p = \sqrt{\frac{\lambda_s + 2\mu_s}{\rho_s}} \quad (6)$$

(λ_s and μ_s are the Lamé parameters of structure material)

- ▶ The thickness parameter h_{sn} depends only on time step size and structure material properties

New Robin-Neumann strongly coupled scheme

Discretization of pressure equation at the interface

$$\left(\frac{1}{a_P}\right)_i \mathbf{n}_i \cdot (\nabla p)_i = \mathbf{S}_i^o \cdot \mathbf{v}_i^o \quad (7)$$

$$\left(\frac{1}{a_P}\right)_i = \Delta t \quad (8)$$

$$\mathbf{S}_i^n \cdot \mathbf{v}_i^n = \mathbf{S}_i^o \cdot \mathbf{v}_i^o - \left(\frac{1}{a_P}\right)_i \mathbf{n}_i \cdot (\nabla p)_i \quad (9)$$

Boundary condition for velocity at the interface

- ▶ Tangential component of velocity specified (Dirichlet BC)
- ▶ Normal component of velocity calculated using flux obtained after solution of pressure equation (flux-corrected velocity)

$$\mathbf{v}_{f,i} = (\mathbf{I} - \mathbf{n}_i \mathbf{n}_i) \cdot \mathbf{v}_{s,i} + \mathbf{n}_i (\mathbf{S}_i^n \cdot \mathbf{v}_i^n) / S_i^n \quad (10)$$

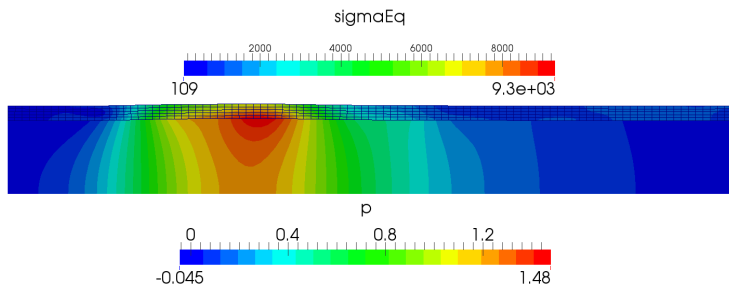
New Robin-Neumann strongly coupled scheme

Proposed solution procedure:

- 1: Predict interface displacement: solve solid model using fluid force from the previous time step, transpose displacements and acceleration from the solid to the fluid side of the interface, solve mesh motion equation, move fluid mesh
- 2: Solve fluid model using proposed Robin BC for the pressure at the interface
- 3: Transpose force from the fluid to the solid side of the interface and solve solid model
- 4: Transpose displacements and acceleration from the solid to the fluid side of the interface, solve mesh motion equation
- 5: Calculate interface residual \mathbf{r}
- 6: **if** $|\mathbf{r}| > \epsilon$ **then**
| Return to step 2
end

Pressure pulse propagation through thick elastic pipe

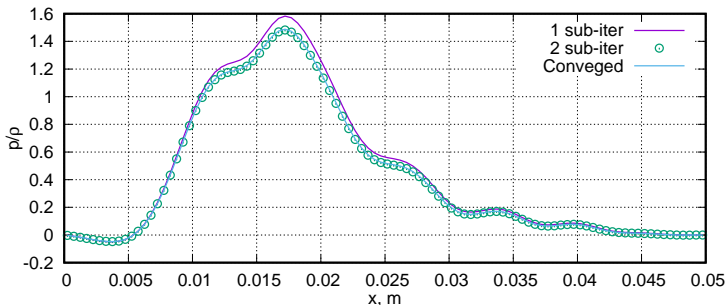
- ▶ Dimensions: $r = 0.005$ m, $L = 0.05$ m, $\delta = 0.001$ m
- ▶ Fluid: $\rho_f = 1000$ kg/m³, $\mu_f = 0.003$ m²/s
- ▶ Solid: $\rho_s = 1200$ kg/m³, $E = 3 \times 10^5$ N/m², $\nu = 0.3$
- ▶ Calculated as axisymmetric case with time step size $\Delta t = 2.5 \times 10^{-5}$ s and $h_{sn} = 0.5 \times 10^{-3}$ m
- ▶ During first 0.003 s, an overpressure of 1333.2 Pa is applied at the inlet.



Pressure pulse propagation through thick elastic pipe

Solution obtained using Robin-Neumann coupling procedure with 1 and 2 sub-iterations is compared with the corresponding fully converged solution

Pressure along the interface at the time instance $t = 0.005$ s



Pressure pulse propagation through thick elastic pipe

Table: Number of sub-iterations per time step and relative duration with respect to the RN scheme for the propagation of pressure pulse through thick elastic pipe.

Algorithm	Iterations	Duration
RN	3	1
DN/IQN-ILS	17	4.8
DN/Aitken	33	7.4

Conclusions

- ▶ Proposed R-N coupling approach is stable even if only one sub-iteration is used after solid predictor step (loosely coupled)
- ▶ Results of such loosely coupled scheme compares very well with the corresponding strong coupling results
- ▶ According to presented results, procedure used to calculate thickness parameter h_{sn} seems to be appropriate (time step size is selected in such a way so that h_{sn} is smaller than maximal structure thickness)
- ▶ Next steps:
 - ▶ Implementation of Robin BC for tangential component of fluid velocity and solid side of the interface
 - ▶ Finalize implementation of kinematically coupled β -scheme by introduction of coupling via initial condition at the interface